Rossmoyne SHS Mathematics Department	MATHEMATICS SPECIALIST 3CD	Semester 1 2011 EXAMINATION

NAME: SOLUTIONS

TEACHER: Mrs Benko Mr Birrell Ms Robinson

Section One: Calculator-free

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

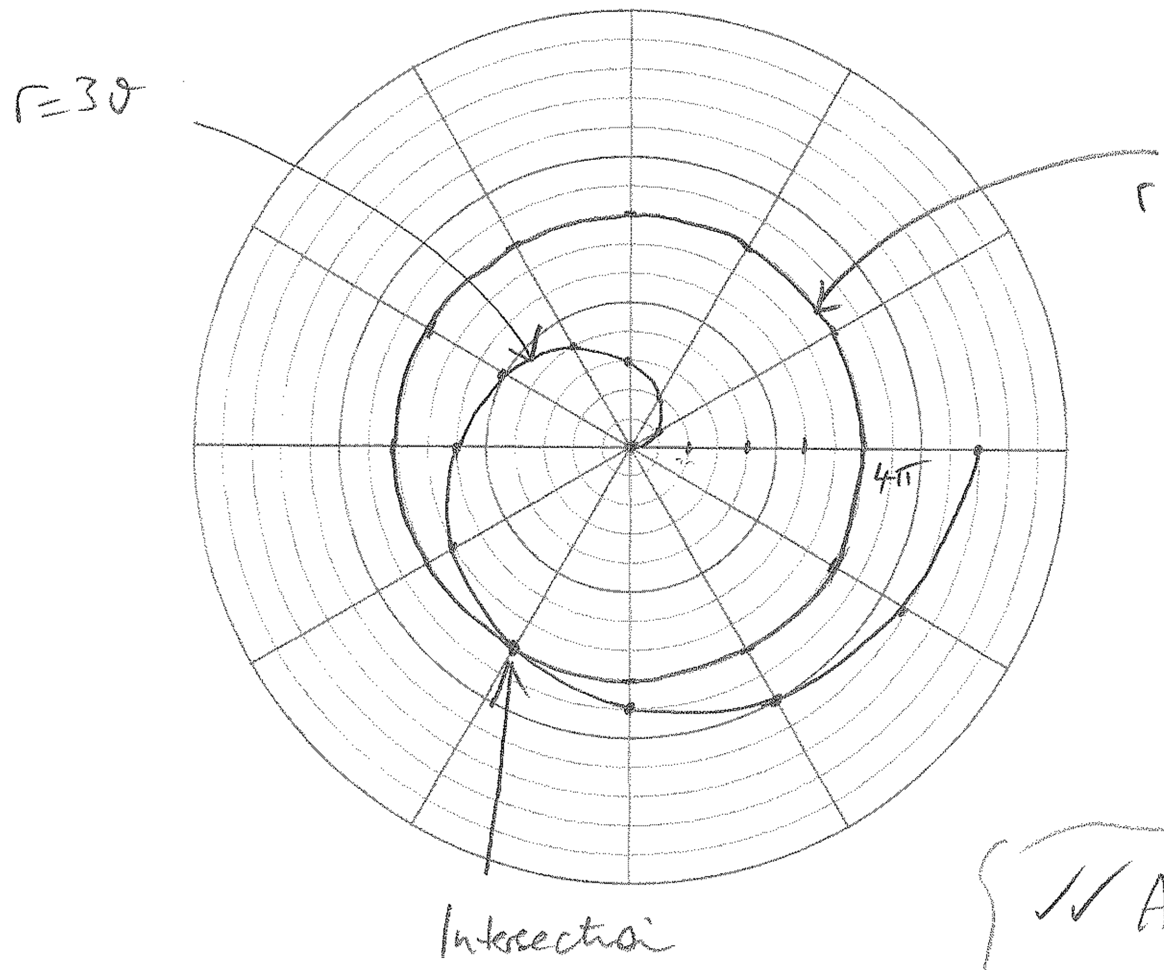
To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

1. (3, 1 = 4 marks)

(a) On the axes below, draw the graphs of $r = 3\theta$ and $r = 4\pi$, for $0 \le \theta \le 2\pi$.



✓✓ Accuracy
✓ Shape

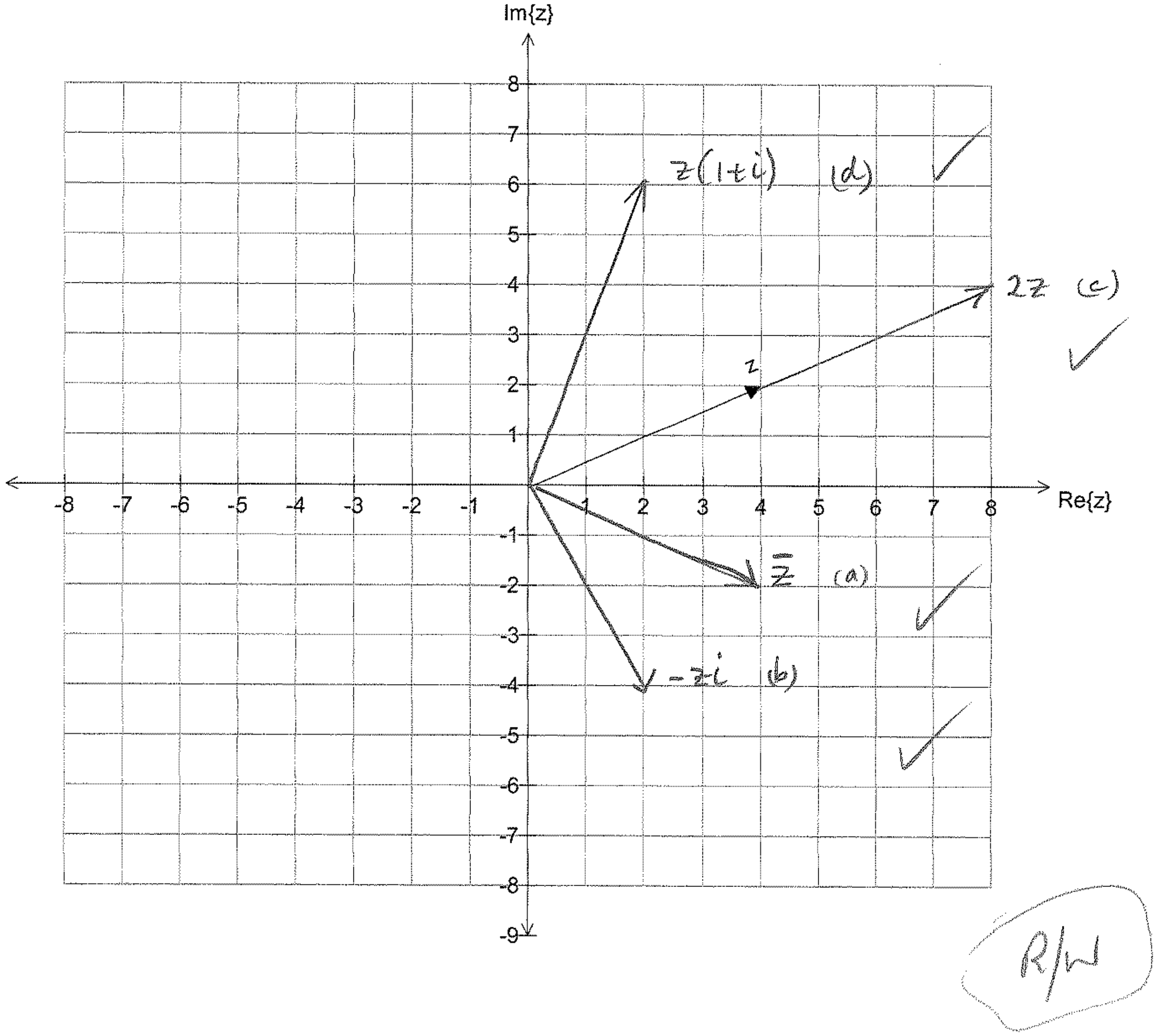
(b). Determine the exact point of intersection of these two graphs in polar form.

$\left(4\pi, \frac{4\pi}{3}\right)$

or

$\left(4\pi, -\frac{2\pi}{3}\right)$ ✓

2. (1, 1, 1, 1 = 4 marks)
The complex number z is shown on the Argand diagram below.



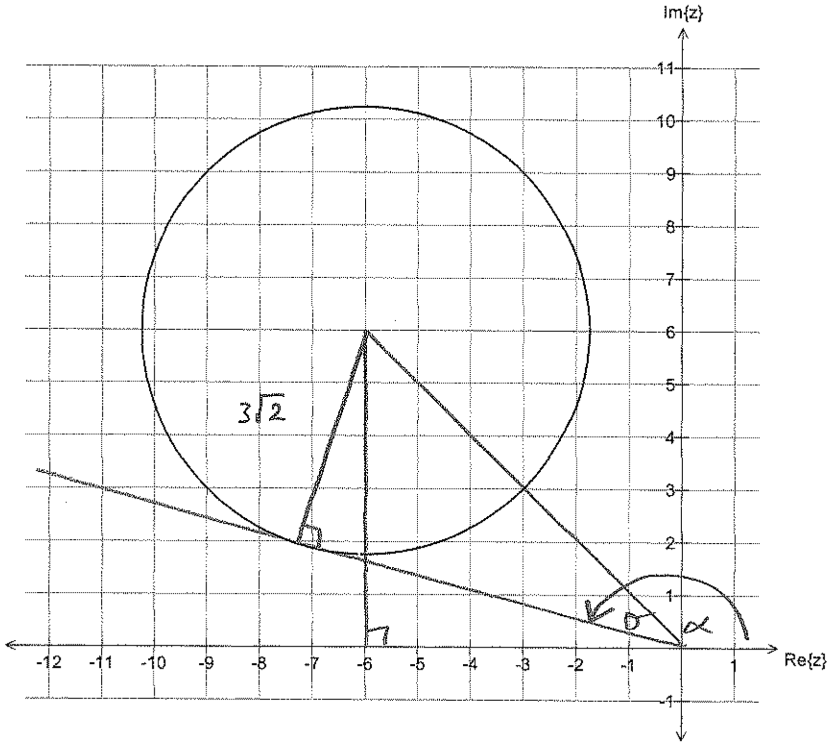
Show the following as vectors on the Argand diagram above.

(a) $\bar{z}$ (b) $-zi$
(c) $2z$ (d) $z(1+i)$

4

3. (1, 3 = 4 marks)

The locus of a complex number z is a circle of radius $3\sqrt{2}$ units as shown below.



(a) Express the locus mathematically in terms of z.

$|z-(-6+6i)| = 3\sqrt{2}$ ✓

or

$|z+6-6i| = 3\sqrt{2}$

R/W

(b) Determine the maximum value of Arg(z).

4

$\alpha = \pi - \tan^{-1}(1)$

$\sin\theta = \dfrac{3\sqrt{2}}{|-6+6i|} = \dfrac{3\sqrt{2}}{\sqrt{72}} = \dfrac{3\sqrt{2}}{6\sqrt{2}} = \dfrac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \pi/6$

max Arg(z)

$= \alpha + \theta$ ✓

$= \pi - \tan^{-1}(1) + \sin^{-1}\left(\frac{1}{2}\right)$ ✓

$= \pi - \dfrac{\pi}{4} + \dfrac{\pi}{6}$

$= \dfrac{11\pi}{12}$. ✓

(or $165°$)

4. (3 marks)
Give the vector equation of the line that is perpendicular to the plane with equation $\mathbf{r}.(6\mathbf{i} - \mathbf{j} - \mathrm{k}) = 12$ and containing the z intercept of the plane.

z - intercept is $(0, 0, -12)$

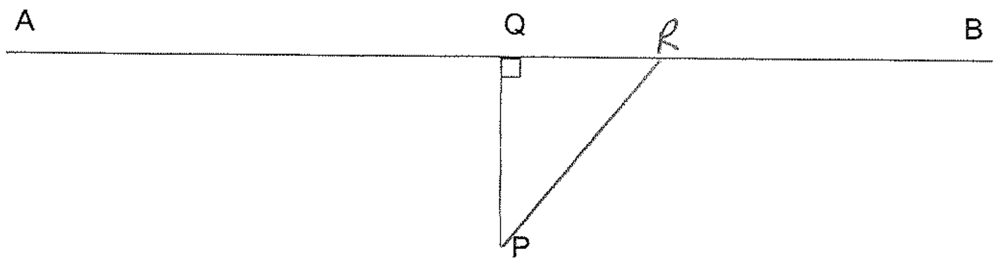
$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ -12 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$$

eq^{n} with λ

$$\underset{\sim}{r} = \begin{pmatrix} 6\lambda \\ -\lambda \\ -12 - \lambda \end{pmatrix}$$

$\boxed{3}$

5. (4 marks)
Point P is a point in the plane not on $\overline{AB}$. Let Q be the point such that $\overline{PQ}$ is perpendicular to $\overline{AB}$.



Prove, using the method of contradiction, that the shortest distance from point P to $\overline{AB}$ is the distance PQ.

$\boxed{4}$

Assume that perp. dist PQ is NOT the shortest distance. ie. Let R be some point on $\overline{AB}$ such that PR < PQ

In right $\triangle QPR$, PR is hyp., and so is longest side in triangle. $\therefore$ PR > PQ

But this contradicts the assumption PR < PQ

$\therefore$ Assumption that PQ is not shortest side must be incorrect.

$\therefore$ PQ (the perp.) is the shortest side.

6. (3, 3 = 6 marks)
Differentiate the following with respect to x. Do not simplify.

(a) $y = \sqrt{2x^3}\sin\dfrac{x}{3}$

$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{2x^3}}\sin\left(\frac{x}{3}\right) + \sqrt{2x^3}\cos\left(\frac{x}{3}\right)\cdot\left(\frac{1}{3}\right)$$

$$\text{ie } \frac{dy}{dx} = \frac{1}{2}(2x^3)^{-1/2}(6x^2)\sin\left(\frac{x}{3}\right) + \sqrt{2x^3}\cos\left(\frac{x}{3}\right)\cdot\left(\frac{1}{3}\right)$$

(b) $y = 5\cos^4(x^2 - 3x)$

$$\frac{dy}{dx} = 20\cos^3(x^2-3x)\cdot(-\sin(x^2-3x))\cdot(2x-3)$$

6

6

7. (~~5~~ marks)
Determine the derivative of f(x) = cos (5x) by first principles.

i.e. using $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\cos(5x + 5h) - \cos(5x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(5x)\cos(5h) - \sin(5x)\sin(5h) - \cos(5x)}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{5\cos(5x)(\cos(5h) - 1)}{5h}}_{=0} - 5\sin(5x) \cdot \underbrace{\frac{\sin 5h}{5h}}_{=1}$$

$$= \underline{-5\sin 5x}$$

$\boxed{5}$

8. (3 marks)
Prove the following trigonometric identity.

$$\tan^2 x \equiv \frac{1-\cos 2x}{1+\cos 2x}$$

$$RHS = \frac{1 - \cos 2x}{1 + \cos 2x}$$

$$= \frac{1 - (1 - 2\sin^2\theta)}{1 + (2\cos^2\theta - 1)}$$

$$= \frac{\cancel{2}\sin^2\theta}{\cancel{2}\cos^2\theta}$$

$$= \tan^2\theta$$

$$= LHS$$

3

9. (2, 4 = 6 marks)

Function f is defined by $f(x) = a^{\log_b x}$ for $x > 0$ where a, b are positive real constants.

(a) Show clearly that f(x) can be written in the form $e^{\frac{(\ln a)(\ln x)}{\ln b}}$.

$$\ln f(x) = (\log_b x) \ln a$$

$$f(x) = e^{(\log_b x)(\ln a)} \checkmark$$

$$f(x) = e^{\frac{(\ln x)(\ln a)}{(\ln b)}} \checkmark$$

(b) Hence, using the expression from part (a), determine $\int_e^{e^2} \frac{a^{\log_b x}}{x} dx$.

$$\int_e^{e^2} \frac{e^{\frac{(\ln x)(\ln a)}{(\ln b)}}}{x} dx \checkmark$$

$$= \left. \frac{(\ln b)}{(\ln a)} e^{\frac{(\ln a)(\ln x)}{(\ln b)}} \right]_e^{e^2} \checkmark$$

$$= \frac{\ln b}{\ln a}\left[e^{\frac{2 \ln a}{\ln b}} - e^{\frac{\ln a}{\ln b}} \right]$$

✓ ✓

$\boxed{6}$